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(h_0, h, M_0)-Stability for Integro-Differential Equations

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INTRODUCTION

The advantage of studying the stability properties of differential equations by means of two different measures and the generality and unification which result by such an approach is sufficiently well known [4, 5].

Quite recently, the concept of M_0-stability is developed to describe a general type of invariant set and its stability behavior [6]. This notion is a natural generalization of the usual concepts of eventual stability and the stability of asymptotically self-invariant sets [2].

The objective of this paper is to introduce a very general type of stability called "(h_0, h, M_0)-stability" by combining the above two notions of stability and study these new stability properties relative to system of integro-differential equations of Volterra type. This new notion of stability allows us to consider the initial values on surfaces that crucially depend on the initial time and also to introduce different topologies in the definition of stability. Our approach here is to reduce the study of an integro-differential system into an ordinary differential equation and thus basically depends on choosing appropriate minimal subsets of a suitable space of continuous functions [1, 3], along which the time derivative of the Liapunov function is estimated.

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1. Preliminaries

Consider the initial value problem

$$x'(t) = f(t, x(t)) + \int_{t_0}^{t} g(t, s, x(s))\, ds, \qquad x(t_0) = \psi(t_0, x^*), \tag{1.1}$$

where $f, \psi \in C[R_+ \times R^n, R^n]$ and $g \in C[R_+ \times R_+ \times R^n, R^n]$. Let $x(t) = x(t, t_0, \psi(t_0, x^*))$ be any solution of (1.1) existing for all $t \geqslant t_0 \geqslant 0$.

Let $M = M(R_+, R^n)$ be the space of all measurable mappings from R_+ to R^n such that $x \in M$ if and only if $x(t)$ is locally integrable and

$$\sup_{t>0} \int_t^{t+1} \|x(s)\|\, ds < \infty.$$

Let $M_0 = M_0(R_+, R^n)$ be a subspace of M consisting of all $x(t)$ such that

$$\int_t^{t+1} \|x(s)\|\, ds \to 0 \qquad \text{as} \quad t \to \infty.$$

The set $S(M_0, \varepsilon)$ is the subset of M defined by

$$S(M_0, \varepsilon) = \left\{ x \in M \colon \lim_{t \to \infty} \sup \int_t^{t+1} \|x(s)\|\, ds \leqslant \varepsilon \right\}.$$

Let $h, h_0 \in C[R_+ \times R^n, R_+]$ and $\inf\{h_0(t, x) \colon (t, x) \in R_+ \times R^n\} = 0$. We shall assume that the measure h_0 is uniformly finer than the measure h, that is, there exists a constant $\lambda > 0$ and a function $\phi \in K$ (see Definition 2.4) such that $h_0(t, x) < \lambda$ implies $h(t, x) < \phi(h_0(t, x))$. For the sake of convenience, let us state the following definitions.

Definition 1.1. Let $A \in R^n$. A is (h_0, h, M_0)-invariant relative to system (1.1) if $x^* \in A$ and $h_0(s, \psi(s, x^*)) \in M_0$, then $h(\cdot, x(\cdot, s, \psi(s, x^*)) \in M_0$.

Definition 1.2. Relative to system (1.1), the set A is said to be

(d_1) (h_0, h, M_0)-stable if for each $\varepsilon > 0$ there exist $\tau_1(\varepsilon)$, $\tau_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $\delta_1(t_0, \varepsilon)$, $\delta_2(t_0, \varepsilon)$ such that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)))\, ds < \varepsilon \qquad \text{for all} \quad t \geqslant t_0 + 1$$

whenever $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*))\, ds < \delta_2$, $t_0 \geqslant \tau_1(\varepsilon)$.

(d_2) (h_0, h, M_0)-uniformly stable if (d_1) holds with δ_1 and δ_2 being independent of t_0.

(d$_3$) (h_0, h, M_0)-equiasymptotically stable if (d$_1$) holds and for any $\eta > 0$ there exist positive numbers $\hat{\delta}_1(t_0)$, $\hat{\delta}_2(t_0)$, τ_0, and $T = T(t_0, \eta)$ such that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*))\, ds < \eta \qquad \text{for all} \quad t \geqslant t_0 + 1 + T,\ t_0 \geqslant \tau_0$$

whenever $x^* \in S(A, \hat{\delta}_1)$ and $h_0(s, \psi(s, x^*)) \in S(M_0, \hat{\delta}_2)$.

(d$_4$) (h_0, h, M_0)-uniformly asymptotically stable if (d$_2$) and (d$_3$) hold with numbers $\hat{\delta}_1$, $\hat{\delta}_2$ and T in (d$_3$) being independent of t_0.

Remark 1.3. We know that (h_0, h)-stability [4, 5] properties include several type of stabilities in the usual sense. Also, the M_0-stability [6] is much more general than eventual stability and the stability of asymptotically self-invariant sets. Consequently, it is not difficult to observe that our (h_0, h, M_0)-stability notion covers a very broad class of stability concepts. We give below a couple of special cases to demonstrate the generality of the new concept (h_0, h, M_0)-stability:

(i) If $h_0(t, x) = h(t, x) = \|x\|$, then (d$_1$) implies M_0-stability of the set A relative to (1.1).

(ii) If $h_0(t, x) = \|x\|$ and $h(t, x) = \|x\|_k = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}$, $k < n$, then (d$_1$) implies M_0-partial stability of the set A relative to (1.1).

Definition 1.4. A function a is said to belong to

(i) the class K if $a \in C[R_+, R_+]$, $a(0) = 0$ and $a(r)$ is strictly increasing in r;

(ii) the class KC if $a \in K$ and a is convex.

Definition 1.5. h_0 has the property P if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $x^* \in A$ and $\psi(s, x^*) \in S(M_0, \delta)$, then $b(h_0(s, \psi(s, x^*)) \in S(M_0, \varepsilon)$ where $b \in K$.

Definition 1.6. For any $h \in C[R_+ \times R^n, R_+]$, let

$$Q(h, \rho) = \{(t, x) \in R_+ \times R^n : h(t, x) < \rho\}.$$

The scalar function $V \in C[R_+ \times R^n, R_+]$ is said to be

(i) h-positive definite if there exists a function $a \in K$ such that

$$a(h(t, x)) \leqslant V(t, x), \qquad (t, x) \in Q(h, \rho).$$

(ii) h_0-decrescent if there exists a $\rho_0 > 0$ and function $b \in K$ such that $h_0(t, x) < \rho_0$ implies

$$V(t, x) \leqslant b(h_0(t, x)), \qquad (t, x) \in Q(h_0, \rho_0).$$

Note that $\rho_0 \in (0, \lambda)$ is such that $b(\rho_0) < \rho$.

Let us now list the following assumptions before we proceed further.

(H_0) $V \in C[R_+ \times R^n, R_+]$, V is locally Lipchitzian in x, and

(i) $a(h(t, x)) \leqslant V(t, x)$, $(t, x) \in Q(h, \rho)$,

(ii) $V(t, x) \leqslant V(h_0(t, x))$, $(t, x) \in Q(h_0, \rho_0)$,

where $a, b \in K$.

(H_1) $g_0, g \in C[R_+ \times R_+, R]$; $\phi, \hat{\phi} \in C[R_+ \times R_+, R_+]$, $g_0(t, u) \leqslant g(t, u)$, $r(t, t_0, \phi(t_0, u^*))$ is the right maximal solution of

$$u' = g(t, u), \qquad u(t_0) = \phi(t_0, u^*), \tag{1.2}$$

on $[t_0, \infty)$ and $\eta(t, t^0, \hat{\phi}(t^0, v^*))$ is the left maximal solution of

$$v' = g_0(t, v), \qquad v(t^0) = \hat{\phi}(t^0, v^*) \tag{1.3}$$

existing on $t_0 \leqslant t \leqslant t^0$.

(H_2) $D_- V(t, x) = \lim_{h \to 0} \inf (1/h)[V(t, x + h(f(t, x) + \int_{t_0}^t g(t, s, x(s))\, ds) - V(t, x)] \leqslant g(t, V(t, x))$ on Ω, where

$$\Omega = \{x \in C[R_+, R^n]: V(s, x(s)) \leqslant \eta(s, t, V(t, x(t)), t_0 \leqslant s \leqslant t\}.$$

We need the following known results ([1, 6, 7]) in our subsequent discussions.

THEOREM 1.1. Suppose (H_1) *holds with* $r(t^0, t_0, \phi(t_0, u^*)) = \hat{\phi}(t_0, u^*)$, *then*

$$r(t, t_0, \phi(t_0, u^*)) \leqslant \eta(t, t^0, \hat{\phi}(t_0, u^*)) \qquad \textit{for} \quad t_0 \leqslant t \leqslant t^0.$$

THEOREM 1.2. *Let* (H_0), (H_1), *and* (H_2) *hold. Let* $x(t, t_0, \psi(t_0, x^*))$ *be any solution of* (1.1) *such that* $V(t_0, \psi(t_0, x^*)) \leqslant \phi(t_0, u^*)$. *Then*

$$V(t, x(t, t_0, \psi(t_0, x^*)) \leqslant r(t, t_0, \phi(t_0, u^*)) \qquad \textit{for all} \quad t \geqslant t_0.$$

THEOREM 1.3. *If a is a convex function and f is integrable on R_+, then*

$$a\left(\int_0^t f(s)\, ds\right) \leqslant \int_0^t a(f(s))\, ds.$$

Finally, we now define the concepts analogous to (d_1) to (d_4) for the comparison equation (1.2). The set $u=0$, relative to (1.2) is M_0-invariant if whenever $\phi(s, 0) \in M_0$, then $u(\cdot, s, \phi(s, 0)) \in M_0$. The set $u=0$ is said to be with respect to Eq. (1.2),

(d_i^*) M_0-stable if for each $\varepsilon > 0$ there exist $\tau_1(\varepsilon)$ and $\delta_1(t_0, \varepsilon)$, $\delta_2(t_0, \varepsilon)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*))\, ds < \varepsilon, \ t \geqslant t_0 + 1,$$

whenever $u^* < \delta_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\, ds < \delta_2$, $t_0 \geqslant \tau_1(\varepsilon)$.

The remaining notions (d_2^*)–(d_4^*) corresponding to (d_2)–(d_4) can be readily formulated.

2. Main Results

In this section we shall give sufficient conditions for (h_0, h, M_0)-stability of integro-differential system (1.1) in terms of Liapunov functions.

THEOREM 2.1. *Assume that*

(i) (H_0), (H_1), *and* (H_2) *hold*;

(ii) $a \in KC$, $b \in K$, *and* h_0 *has the property* P;

(iii) h_0 *is uniformly finer than* h.

Then, M_0*-stability properties of the set* $u=0$ *relative to* (1.2) *imply the corresponding* (h_0, h, M_0)*-stability properties of the set* A *relative to the system* (1.1).

Proof. We shall only give the proof for (h_0, h, M_0)-uniform stability property of the set A relative to the system (1.1) which will indicate the interplay between several concepts employed. The proof of other stability properties can be constructed on similar lines.

Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given. Suppose the set $u=0$ is M_0-uniformly stable with respect to (1.2). Then, given $a(\varepsilon) > 0$, there exist $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, and $\tau_1(\varepsilon)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*))\, ds < a(\varepsilon), \qquad t \geqslant t_0 + 1, \tag{2.1}$$

whenever $0 \leqslant u^* < \delta_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\, ds < \delta_2$, $t_0 \geqslant \tau_1(\varepsilon)$.

In view of assumptions (ii) and (iii), and the definition of the set A, we can find $\hat{\delta}_1(\varepsilon)$, $\hat{\delta}_2(\varepsilon)$, and $\tau_2(\varepsilon)$,

$$\int_{t_0}^{t_0+1} b(h_0(s, \psi(s, x^*)) \, ds < \delta_2, \qquad t_0 \geqslant \tau_2(\varepsilon)$$

and $x^* \in S(A, \hat{\delta}_1)$, $\int_{t_0}^{t_0+1} h_0(s, \psi(s, x^*)) < \hat{\delta}_2$.

Let $\delta_1^*(\varepsilon) = \min(\delta_1, \hat{\delta}_1)$, $\delta_2^*(\varepsilon) = \min(\delta_2, \hat{\delta}_2)$, and $\tau^*(\varepsilon) = \max(\tau_1, \tau_2)$. Choose x^* such that $x^* \in S(A, \delta_1^*)$ and $\int_{t_0}^{t_0+1} b(h_0(s, \psi(s, x^*)) \, ds < \delta_2^*$ for $t_0 \geqslant \tau^*(\varepsilon)$. Then we claim that

$$\int_{t_0}^{t_0+1} h(t, x(t, s, \psi(s, x^*)) \, ds < \varepsilon.$$

If this is not true, then there is a first $t_1 > t_0 + 1$, $t_0 \geqslant \tau^*(\varepsilon)$ such that

$$\int_{t_0}^{t_0+1} h(t_1, x(t_1, s, \psi(s, x^*))) \, ds = \varepsilon$$

and

$$\int_{t_0}^{t_0+1} h(t, x(t, x, \psi(s, x^*)) \, ds < \varepsilon \qquad \text{for} \quad t_0 + 1 \leqslant t < t_1$$

and $t_0 \geqslant \tau^*(\varepsilon)$.

Let $r(t, s, \phi(s, u^*))$ be the maximal solution of (1.2). Set $u^* = d(A, x^*)$. Then $V(s, \psi(s, x^*)) \leqslant b(h_0(s, \psi(s, x^*)) \equiv \phi(s, d(A, x^*)) = \phi(s, u^*)$ and hence by Theorem 1.2, it follows that

$$V(t, x(t, s, \psi(s, x^*)) \leqslant r(t, s, \phi(s, u^*)), \qquad t_0 \leqslant t \leqslant t_1. \tag{2.2}$$

Since $u^* < \delta_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*) \, ds < a(\varepsilon)$, the assumption (ii), (2.1), (2.2), and Theorem 1.3 lead to

$$\begin{aligned} a(\varepsilon) &\leqslant a\left(\int_{t_0}^{t_0+1} h(t_1, x(t_1, s, \psi(s, x^*)) \, ds\right) \\ &\leqslant \int_{t_0}^{t_0+1} a(h(t_1, x(t_1, s, \psi(s, x^*))) \, ds \\ &\leqslant \int_{t_0}^{t_0+1} V(t_1, x(t_1, s, \psi(s, x^*))) \, ds \\ &\leqslant \int_{t_0}^{t_0+1} r(t_1, s, \phi(s, u^*)) \, ds < a(\varepsilon), \end{aligned}$$

a contradiction. Hence the set A relative to (1.1) is (h_0, h, M_0)-uniformly stable. This completes the proof.

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